Quantum control without access to the controlling interaction

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In our model a fixed Hamiltonian acts on the joint Hilbert space of a quantum system and its controller. We show under which conditions measurements, state preparations, and unitary implementations on the system can be performed by quantum operations on the controller only.

It turns out that a measurement of the observable A and an implementation of the one-parameter group $exp(iAr)$ can be performed by almost the same sequence of control operations. Furthermore measurement procedures for $A+B$, for $(AB+BA)$, and for $i[A,B]$ can be constructed from measurements of A and B. This shows that the *algebraic* structure of the set of observables can be explained by the *Lie group* structure of the unitary evolutions on the joint Hilbert space of the measuring device and the measured system.

A spin chain model with nearest neighborhood coupling shows that the border line between controller and system can be shifted consistently.

I. INTRODUCTION

Quantum state control might be described as the preparation, manipulation and measurement of quantum systems. In modern research, especially in the field of quantum optics and quantum communication, the control of simple quantum systems is one of the main goals. Typical problems are the manipulation of the inner degree of an ion in a trap, the polarization of a photon or the preparation of Fock states of light. All these manipulations of quantum systems are performed by the interaction between the controlled system and the particles or fields surrounding it. Quantum state control can be formulated more precisely as follows: the pure states of the quantum system to be controlled are described by the one-dimensional subspaces of a Hilbert space $\mathcal{H}_s$. For simplicity we will assume $\mathcal{H}_s$ to be finite-dimensional. Mainly, there are three actions on a quantum system which one wishes to perform:

1. Initialization of a certain pure state: let ρ be an arbitrary density matrix on $\mathcal{H}_s$ and $|\psi\rangle \in \mathcal{H}_s$ be the vector of the state which should be prepared. Then initialization consists in causing the transition $\rho \mapsto |\psi\rangle\langle\psi|$.

2. Unitary control: the vector state $|\psi\rangle$ is transformed to $u|\psi\rangle$, a density matrix ρ to $u\rho u^\dagger$ for any arbitrary unitary matrix u.

3. Ideal quantum-non-demolition (QND) measurement: Let $(P_i)_{i\in I}$ be a complete orthogonal family of projections (in the sequel referred to as 'the observable (P_i)'). Let $\mathcal{H}_c$ be the Hilbert space of any system acting as the measurement apparatus ('c' for 'controller', as measurement apparatus and controller are in the following assumed to be given by the same system). A measurement of the observable (P_i) is a unitary evolution starting in a product state $|\phi\rangle\otimes|\psi\rangle \in \mathcal{H}_c\otimes\mathcal{H}_s$ and ending up in the state $\sum_i |\phi_i\rangle \otimes P_i|\psi\rangle$ where $(|\phi_i\rangle)_{i\in I}$ is an orthonormal family of states of the measurement apparatus ('the pointer positions'). As usual, a 'measurement for the observable A' for any self-adjoint operator A is defined by the family of its spectral projections.

Initialization can be performed by a unitary control operation which is conditioned on the result of a former measurement. Hence we shall only deal with measurements and unitary control operations. Furthermore we shall not care about more general types of measurements (positive operator valued measurements) or more general quantum operations (completely positive maps), since they can be obtained from measurements and unitary operations on larger systems (see [1]) by restriction to the considered system.

Common approaches to quantum control theory [2,4] are based on the assumption that the controlling person is able to change the interaction between the system and its controller. This means that one can manipulate a (possibly time-dependent) Hamiltonian H acting on $\mathcal{H}_c\otimes\mathcal{H}_s$ or even the system's free Hamiltonian acting on $\mathcal{H}_s$ by extern access; sometimes it is assumed that one is able to control at least a small perturbation of the Hamiltonian [2]. Of course such an approach is well justified in many realistic physical situations: macroscopic fields should rather be considered as classical parameters determining the system's Hamiltonian than as an interaction Hamiltonian between system and its environment.

Our motivation for investigating models with no classical control parameters is twofold:

1. Our model can describe many physical situations where one component of a larger quantum system is easier to access than the other parts: imagine a

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molecule where a given control mechanism for one atom should be used for controlling the other sort of atoms indirectly. Another example is given by several experiments of quantum optics: consider an atom in a cavity controlled by the field mode. Manipulating the field in order to control the atom is clearly an indirect control operation in the sense discussed below.

2. Our model shows a symmetry between *measurement procedures* and *unitary implementations* which is covered up in the common approach: consider a unitary evolution u on the joint Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_s$ that can be decomposed into a sum

$$u = \sum_j v_j \otimes P_j$$

where the operators P_j are mutual orthogonal projections and v_j are sufficiently distinct unitary transformations on the controller. Then u can be used for measuring an observables with spectral projections P_j. By exchanging the role of controller and system one obtains a control mechanism for the system by switching between states in the image of different projections P_j.

For a given Hamiltonian we address the question for the set of possible *measurements* and the set of possible *unitary implementations*. We show that the answers to both questions are given by a common theory. This is an important justification of our approach.

One might argue that our approach to the problem of quantum control only shifts the problem to the question of how to manipulate the controller's quantum state. We investigate the mathematical problems arising if such a controller is controlled by a meta-controller and give conditions under which this is possible. We do not discuss the question of how to control the last controller.

Of course this problem is an extension of the problem of quantum mechanical measurements ('who measures the measuring device?', compare [5–7]). Having in mind the connection to those never-ending debates, in the sequel we will be content with the result that the borderline between system and controller (so to speak the 'Heisenberg cut of quantum state control') can be shifted consistently. We are aware of the fact that this shift causes large unsolved philosophical, physical or biological problems, e.g., the problem of the interface between micro- and macrophysics[1] (if one takes a macroscopic system as the controller), or even the problem of the *freedom of will* (the brain as the controller). Although we ignore those problems in the sequel, the results presented below give new insights into the structure of the set of measurable observables and implementable transformations.

Our approach does not refer to any assumptions on physical properties of the system and the interaction. It does not even assume that the system is a particle. It might also be one degree of freedom of a particle or any artificial decomposition of a particle's Hilbert space to a tensor product of two unphysical components. The results presented below should therefore be considered as a structure theory of input, manipulation and readout of quantum information. The assumption that every dynamical evolution is caused by the Hamiltonian represents the only physical input in such an information theory of quantum state control.

The theory shows that the possibility of performing every kind of measurement is equivalent to the possibility of performing every kind of unitary transformation, since the mathematical conditions on the form of the interaction Hamiltonian coincide.

II. THE INTERFACE GROUP

We describe the quantum system and its controller by the Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_c$, respectively. For simplicity we assume both spaces to be of finite dimension. Let W be the group of special unitary transformations acting on $\mathcal{H}_c$ considered canonically as a subgroup of the group of unitary transformations on $\mathcal{H}_c \otimes \mathcal{H}_s$. We assume that all that the experimentalist can do is performing a procedure of the following form: 'perform a transformation $w_1 \in W$, wait the time t_1, perform $w_2 \in W$, ..., perform $w_n \in W$, wait the time t_n.' Formally, we denote such a procedure by

$$p := (t_1, w_1, \ldots, t_n, w_n) \text{ with } t_i \geq 0, w_i \in W.$$

We denote the set of such procedures by P. The procedure p implements the unitary

$$u_p := w_n e^{iHt_n} \ldots w_1 e^{iHt_1},$$

where H is the joint Hamiltonian on $\mathcal{H}_c \otimes \mathcal{H}_s$. Its implementation time t_p is

$$t_p := \sum_j t_j,$$

since we assume that the implementations on the controller can be performed arbitrarily fast.

We consider the set

$$I := \overline{\{u_p \mid p \in P\}}$$

[1] See e.g. [8].

where $\overline{\{\}}$ denotes the closure in the norm topology. Obviously, one has $u_p u_r = u_{rp}$ where rp denotes the procedure obtained by concatenation of the instructions r and p. Since the natural time evolution e^{iHt} is quasiperiodic in finite dimensions, I contains the unitaries e^{iHt} even for negative times and therefore I is a group. We shall refer to I as the *interface group* of the quantum control system $(\mathcal{H}_s, \mathcal{H}_c, H)$. The group I describes the set of possible transformations on the composed system. Unfortunately, the only obvious procedure for implementing e^{iHt} for negative t seems to be given by waiting for a long time until the approximative recurrence of the system. This can take inappropriate long times for large dimensions of $\mathcal{H}_c \otimes \mathcal{H}_s$. A priori, there is no reason why there should be a *fast* realization for e^{-iHt} for small t.

Here we restrict our attention to the particular case where such a fast realization exists, namely that the free Hamiltonian of the system is zero, if the interaction Hamiltonian is written as a sum of tensor products of traceless self-adjoint operators only: the following Lemma is obtained in strong analogy to the first order decoupling technique in [9]:

Lemma 1 Let $H := \sum_j A_j \otimes B_j$ be the Hamiltonian acting on the joint Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_s$. Assume every A_j to be traceless. Let S be a subgroup of W acting irreducibly on $\mathcal{H}_c$. Then

$$\sum_{s \in S} sHs^\dagger = 0.$$

Proof: We have $\sum sHs^\dagger = \sum_i \sum_s sA_i s^\dagger \otimes B_i$. Obviously, each term of the form $\sum_{s\in S} sA_i s^\dagger$ commutes with each $s \in S$. Due to Schur's Lemma the map $A_j \mapsto \sum_s sA_j s^\dagger$ can only map on the multiples of the identity matrix. Since every A_j is traceless, it maps every A_j to zero. □

Hence we have $-H = \sum_{s\in S\setminus\{1\}} sHs^\dagger$. Due to the formula

$$\prod_{s\in S\setminus\{1\}} se^{iH\epsilon}s^\dagger = e^{-iH\epsilon} + O(\epsilon^2),$$

we have found an approximative implementation of the inverse time evolution e^{-iHt} with the implementation time $(|S|-1)|t|$, where $|S|$ is the group order of S.

In the sequel we will assume (if nothing different is said), that the conditions of Lemma 1 are satisfied. This can be justified as follows: decompose an arbitrary Hamiltonian H into a sum

$$H := \sum_j A_j \otimes B_j + A \otimes 1 + 1 \otimes B,$$

with traceless self-adjoint operators A_j, B_j, A, B. The term $1 \otimes B$ can be cancelled by passing to a rotating frame. Apart from this we are able to cancel A by an additional Hamiltonian since we assume to have general access to the controller. In order to work out the structure of the interface group, we use Lie *algebraic* techniques:

Definition 1 Let H be the interaction Hamiltonian acting on the joint Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_s$. Let $\mathcal{W}$ be the set of self-adjoint traceless operators on $\mathcal{H}_c$. Then the *Interface Algebra* $\mathcal{I}$ is the Lie algebra[2] generated by the Lie sub algebra $\mathcal{W}$ and the Hamiltonian H.

Then Lemma 1 allows us to give an operational meaning to the elements of $\mathcal{I}$:

Since unitary operations of the form $e^{\pm iH\epsilon}$ can (approximatively) be implemented during a short time period in the order of ϵ, operations of the form e^{iA} can be implemented with arbitrarily high accuracy if A can be written as sum of multicommutators consisting of H and arbitrary self-adjoint operators on the controller. Those control algorithms can be constructed via the well-known formulas

$$e^{i(A+B)} = \lim_{m\to\infty} (e^{iA/m} e^{iB/m})^m$$

and

$$e^{-[A,B]} = \lim_{m\to\infty} (e^{iA/m} e^{iB/m} e^{-iA/m} e^{-iB/m})^{m^2}. \quad (1)$$

For every A in the interface algebra we can therefore approximate every unitary of the form e^{iAs} with $s \in \mathbb{R}$. We rephrase this by claiming that we can simulate the unitary time evolution corresponding to the 'effective Hamiltonian' A. In order to avoid misinterpretations of this suggestive formulation, we emphasize the following:

- The simulation of the evolution $(e^{iAs})_{s\in\mathbb{R}}$ is discrete: we can choose a small number ϵ and construct a control algorithm implementing $e^{iA\epsilon}$ approximatively. By iteration, we obtain $e^{iA\epsilon n}$ for every $n \in \mathbb{N}$.
- In general the 'simulated time' s does not coincide with the implementation time for e^{iAs}. Assume e.g. $A := i[H, B]$ where B is an operator on $\mathcal{H}_c$. Using the transformations $e^{\pm iH\epsilon}$ and $e^{\pm iB\epsilon}$ we obtain approximatively $e^{[H,B]\epsilon^2}$. This term is of second order in the simulation time since the latter is proportional to ϵ. Using such a scheme, the running time for the simulation goes to infinity for increasing accuracy.

[2]We do not claim, that $\mathcal{I}$ is the Lie algebra of the Lie group I, since the group can be considerably enlarged by the closure.

The interface algebra $\mathcal{I}$ can be characterized explicitly:

Theorem 1 (Structure of the interface algebra) Set $H := \sum_j A_j \otimes B_j$ where (A_j) and (B_j) are families of traceless self-adjoint operators, linearly independent as $\mathbb{R}$-vectors. Let the dimension of $\mathcal{H}_c$ be larger than 2. Then the interface algebra $\mathcal{I}$ is given by

$$\mathcal{I} = \mathcal{W} \otimes \mathcal{B} + 1 \otimes \mathcal{L} \tag{2}$$

where $\mathcal{B}$ is the self-adjoint part of the C^*-algebra generated by (B_j) and $\mathcal{L}$ is the vector space $\{i[A, B]_{A,B\in\mathcal{B}}\}$.

Proof: Define $\mathcal{D}$ as the set of self-adjoint operators D such that there is an operator $G \in \mathcal{W}$ with the property that $G \otimes D$ is in $\mathcal{I}$. This is equivalent to the statement that for every $G \in \mathcal{W}$ the operator $G \otimes D$ is an element of $\mathcal{I}$ since every $\mathbb{R}$-linear map on $\mathcal{W}$ can be generated by concatenations and sums of maps of the form $i[A, .]$ with $A \in \mathcal{W}$ due to Lemma 2 in [10]. We show that $D, E \in \mathcal{D}$ implies (a) $DE + ED \in \mathcal{D}$ and (b) $i[D, E] \in \mathcal{D}$: take two arbitrary non-commuting operators $G_1, G_2 \in \mathcal{W}$. Easy calculation shows the equation

$$\begin{aligned} &i[G_1 \otimes D, G_2 \otimes E] + i[G_1 \otimes E, G_2 \otimes D] \\ &= [G_1, G_2] \otimes (DE + ED). \end{aligned}$$

This proves implication (a).

Implication (b) can be seen as follows: choose G_1, G_2 such that G_1^2 and G_2^2 are linearly independent. This is possible for dimension larger than 2. Then

$$i[G_j \otimes D, G_j \otimes E] = G_j^2 \otimes i[D, E] \in \mathcal{I}.$$

Hence

$$(G_1^2 - \lambda G_2^2) \otimes i[D, E] \in \mathcal{I}.$$

Choose $\lambda \in \mathbb{R}$ such that $G_1^2 - \lambda G_2^2$ is traceless and hence an element of $\mathcal{W}$. This proves that $i[D, E]$ is in $\mathcal{D}$. Due to $DE = (DE + ED)/2 - (i/2)i[D, E]$ the complexification $\mathcal{D}+i\mathcal{D}$ of $\mathcal{D}$ is a algebra over $\mathbb{C}$ which is closed under the $(.)^*$-operation. Hence it is a C^*-algebra [11] since it is a subset of a finite dimensional algebra. The following observation shows that $\mathcal{D}$ contains every element B_j: Choose an $\mathbb{R}$-linear map $Y : \mathcal{W} \to \mathcal{W}$ with $Y(A_k) = 0$ for $k \neq j$ and $Y(A_j) = A_j$. By Lemma .. in [10] the operator $(Y \otimes id)(H) = A_j \otimes B_j$ is an element of $\mathcal{I}$. Hence $\mathcal{D}+i\mathcal{D}$ contains the C^*-algebra generated by $(B_j)_j$. This proves that $\mathcal{I}$ contains the set $\mathcal{W} \otimes \mathcal{B}$.

In order to show that $\mathcal{I}$ contains $1 \otimes \mathcal{L}$ we define $\mathcal{F}$ as the set of operators F such that $1 \otimes F$ is an element of $\mathcal{I}$. By choosing $G \in \mathcal{W}$ in such a way that $G^2 = 1$ one can see that $D, E \in \mathcal{D}$ implies that

$$i[D, E] \in \mathcal{F},$$

since $i[G \otimes D, G \otimes E] = G^2 \otimes i[D, E]$.

We have already shown that $\mathcal{I}$ contains the set

$$\mathcal{W} \otimes \mathcal{B} \oplus 1 \otimes \mathcal{L}.$$

Obviously this $\mathbb{R}$-vector space contains $\mathcal{W}$ and H. One checks easily that it is closed under commutators: Let $G_1, G_2 \in \mathcal{W}, A_1, A_2 \in \mathcal{B}, L_1, L_2 \in \mathcal{L}$. Then we obtain:

$$\begin{aligned} &i[G_1 \otimes A_1 + 1 \otimes L_1, G_2 \otimes A_2 + 1 \otimes L_2] \\ &= i[G_1, G_2] \otimes (A_1A_2 + A_2A_1)1/2 \\ &+ (G_1G_2 + G_2G_1) \otimes i[A_1, A_2]1/2 \\ &+ G_1 \otimes [A_1, L_2] + G_2 \otimes [L_1, A_2] + 1 \otimes [L_1, L_2]. \end{aligned}$$

The terms $i[A_j, L_k]$ are clearly in $\mathcal{L}$. $A_1A_2 + A_2A_1$ is an element of $\mathcal{B}$. The operator $G_1G_2 + G_2G_1$ can be written as a linear combination of a traceless part (which is an element of $\mathcal{W}$) and a scalar multiple of the identity. Hence $(G_1G_2 + G_2G_1) \otimes [A_1, A_2]$ is an element of $\mathcal{W} \otimes \mathcal{B} \oplus 1 \otimes \mathcal{L}$. Since the rhs of eq. (2) contains H and $\mathcal{W}$, the proof is complete. □

The assumption $dim(\mathcal{H}_c) \geq 3$ can not be dropped. This could be verified by calculating the interface algebra for $\mathcal{H}_s = \mathcal{H}_c = \mathbb{C}^2$ with

$$H = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y,$$

with the Pauli matrices σ_x an σ_y. This is the x, y-Hamiltonian (as it is often used in solid-states physics [12]). One can show that $\mathcal{I}$ is spanned (as $\mathbb{R}$-vector space) by the 10 basis vectors

$$\sigma_x \otimes \sigma_j,\ \sigma_y \otimes \sigma_j,\ \sigma_z \otimes 1,\ 1 \otimes \sigma_j$$

with $j = x, y, z$. We shall not bother about the case $\mathcal{H}_c = \mathbb{C}^2$. Since we assume to have universal access to the controller we should not only allow unitary operations on the controller but should also take into account unitary operations on the controller coupled to an arbitrary large ancilla system. Such operations might appear as non-unitary operations on the controller in the sense of completely positive trace preserving maps [1], measurements on the controller and so on. Hence it makes sense to embed the original controller Hilbert space into a larger controller space $\tilde{\mathcal{H}}_c := \mathcal{H}_a \otimes \mathcal{H}_c$ with ancilla space $\mathcal{H}_a$ and calculate the interface algebra by taking the canonical extension of the original Hamiltonian H, namely $\tilde{H} := 1 \otimes H$. If we have already $dim(\mathcal{H}_c) \geq 3$ an additional extension does not change the structure of the interface algebra given by Theorem 1 except from the fact that $\mathcal{W}$ is enlarged. There is also another reason for extending the controller Hilbert space in such a way: measurements are only possible if $\mathcal{H}_c$ has sufficiently many 'pointer states'.

Theorem 1 shows that universal unitary control is possible if and only if the operators $\{B_j\}_j$ generate the full algebra of linear maps on $\mathcal{H}_s$. This can be seen as follows: If (B_j) generate the whole algebra then $\mathcal{I}$ contains the

element $D \otimes A$ for every self-adjoint A and every traceless self-adjoint D. Let λ_j be the eigenvalues of D for the eigenvector $|j\rangle$. Due to the equation

$$e^{iD\otimes As}(|j\rangle \otimes |\phi\rangle) = |j\rangle \otimes e^{i\lambda_j As}|\phi\rangle$$

one can implement the unitary group $(e^{iAs})_{s\in\mathbb{R}}$. Assume conversely that the interface algebra contains an operator B enabling the implementation of the unitary group e^{iAs} in the sense that there is an initial state $|\phi\rangle$ of the controller such that

$$e^{iBs}(|\phi\rangle \otimes |\psi\rangle) = |\phi\rangle \otimes e^{iAs}|\psi\rangle,$$

then we have

$$B(|\phi\rangle \otimes |\psi\rangle) = |\phi\rangle \otimes A|\psi\rangle.$$

Without loss of generality assume $|\phi\rangle$ to be the first basis vector of $\mathcal{H}_c$. Define $Z := diag(1, -1, 0, \ldots, 0)$. Consider the $\mathbb{R}$-linear map $L : \mathcal{W} \to \mathcal{W}$ with $L(W) := diag(\langle\psi|W|\psi\rangle, -\langle\psi|W|\psi\rangle, 0, \ldots, 0)$. L is a (non-orthogonal) projection on the $\mathbb{R}$-linear span of Z. Due to Lemma 2 in [10], L can be generated by concatenations and sums of maps of the form $i[W, .]$ with $W \in \mathcal{W}$ and hence $(L \otimes id)(B) =: C$ is in $\mathcal{I}$. Since the image of L is one-dimensional, C is of the form $Z \otimes K$ with an appropriate self-adjoint operator K acting on $\mathcal{H}_s$. We can show that $K = A$ due to the equation

$$C(|\phi\rangle \otimes |\psi\rangle) = |\phi\rangle \otimes A|\psi\rangle.$$

Hence $Z \otimes A$ has to be in $\mathcal{I}$. This implies that A has to be in the C^*-algebra generated by the set $\{B_j\}$. Therefore we have obtained a simple necessary and sufficient condition for the interaction to enable universal unitary control.

III. THE 'HEISENBERG CUT' OF UNITARY CONTROL

The essential result of the investigations above may be rephrased by the statement that indirect quantum control is possible by operations on the controller only (under suitable assumptions on the interaction).

The question why and in which sense control mechanisms for the controller exist stands outside our theory. At first sight, it seems as if the possibility of controlling the controller by a meta-controller can obviously be explained by a mechanism of the same type. It should be emphasized that there is one 'detail' causing problems for doing so: our Lie algebraic approach assumed that the operations on the controller can be implemented arbitrarily fast. This assumption can be justified if the interaction between the meta-controller and the controller is strong compared to the interaction between controller and system. Recalling the fact that control of physical systems becomes more and more a matter of macroscopic forces if we shift the controller towards our muscles (switching a field), this point of view might be justified. If one does not accept such an assumption of 'increasing force', the problem of shifting the 'Heisenberg cut of quantum control' arbitrarily has to be investigated more carefully. We will do this by taking a chain of n finite dimensional quantum systems, where each one is interacting with its 2 nearest neighbors[3]. Than we analyze the possibilities of controlling the n^{th} site by operations on the first one. Note that there is another detail causing difficulties in applying the theory developed so far: The possibility of simulating the inverse time evolution on the joint system $\mathcal{H}_c \otimes \mathcal{H}_s$ by operations on the controller only relied on the fact that the joint Hamiltonian H was assumed to have a decomposition into products of traceless self-adjoint operators. This cannot be the case for a Hamiltonian of the form

$$H := \sum_{j\geq 1} H_{j,j+1}$$

where $H_{j,j+1}$ is the interaction between site j and site $j+1$ since H has clearly a component of the form $1 \otimes A$ where 1 is the identity operator acting on the first site[4].

As already mentioned, there is another possibility for implementing the inverse time evolution which is simply given by waiting. Of course such a technique is only realistic for quantum systems of small dimensions. The following theorem shows that the 'Heisenberg-cut' between the controller and the system to be controlled can be shifted arbitrarily in in a chain of quantum systems with appropriate interactions (provided that one accepts long waiting times for the reccurence of the natural time evolution):

Theorem 2 Let $\bigotimes_{j\leq n} \mathcal{H}_j$ be the Hilbert space of a spin chain of length n with the nearest neighborhood interaction

$$H := \sum_j H_{j,j+1}.$$

Assume every interaction term $H_{j,j+1}$ be of the form

$$H_{j,j+1} = \sum_k A_k^{(j)} B_k^{(j+1)}$$

[3]As an example one might think of a spin chain as in popular models of mathematical solid state physics [13].

[4]Note that it would be confusing here to pass to a rotating frame with respect to the residual Hamiltonian $\sum_{j\geq 2} H_{j,j+1}$ since the corresponding transformation does not preserve the tensor product structure on the remaining chain.

where every $A_k^{(j)}$ is an operator acting on $\mathcal{H}_j$ and $B_k^{(j+1)}$ is an operator acting on $\mathcal{H}_{j+1}$, embedded into the space of the n spins. Let the dimension of each $\mathcal{H}_j$ be greater than 2. For every j let $\{A_k^{(j)}\}_k$ be a set of linearly independent self-adjoint traceless operators. Assume that the set of operators $\{B_k(j)\}_k$ generate the complete algebra $\mathcal{A}_j$ of operators on $\mathcal{H}_j$.

Then the system given by the first m sites $\bigotimes_{j\leq m}\mathcal{H}_j$ can be used as a controller for the remaining $n-m$ sites $\bigotimes_{j>m}\mathcal{H}_j$, i.e., the Lie algebra generated by H and the set of traceless self-adjoint operators $(\bigotimes_{j\leq m}\mathcal{A}_j)^{s.a.tl}$ is the set of self-adjoint traceless operators $(\bigotimes_{j\leq n}\mathcal{A}_j)^{s.a.tl}$ on the Hilbert space of the complete chain.

Proof: (By induction over m) For $m=n$ the statement is trivial. We assume it to be true for $m+1$ and conclude that it is true for m:

Consider the Lie algebra $\mathcal{I}_m$ generated by $(\bigotimes_{j\leq m}\mathcal{A}_j)^{s.a.tl}$ and H. Define L to be the map on $\bigotimes_{j\leq m}\mathcal{A}_j$ annihilating every traceless operator and mapping the identity on itself. By Lemma 2 in [10] $L\otimes id^{\otimes n-m}$ can be generated by operations on the controller, hence $(L\otimes id^{\otimes n-m})(H)=\sum_{m+1\leq j\leq n-1}H_{j,j+1}$ is an element of $\mathcal{I}_m$. Since $\sum_{j\leq m-1}H_{j,j+1}$ is an element of $(\bigotimes_{j\leq m}\mathcal{A}_j)^{s.a.tl}$ we conclude that $H_{m,m+1}$ is in $\mathcal{I}_m$. Due to Theorem 1 every Lie algebra containing $(\bigotimes_{j\leq m}\mathcal{A}_j)^{s.a.tl}$ and $H_{m,m+1}$ contains $\bigotimes_{j\leq m+1}\mathcal{A}_j$. This completes the induction. □

Hence we have proven that arbitrary unitary transformations on the whole chain can be generated by accessing the first site only. In order to measure arbitrary observables of the chain by accessing only the first site it is necessary to couple it to an ancilla quantum system providing enough distinguishable quantum states. With the help of such an ancilla system, every measurements can be performed: we will show in the following section that a controlling interaction which allows universal unitary control allows universal measurements as well.

IV. CORRESPONDENCE BETWEEN MEASUREMENTS AND IMPLEMENTATIONS

In some sense the problem of manipulating a quantum system by extern control mechanisms is the inverse problem to the measurement problem. In the latter one the state of the quantum system has an effect on the measurement apparatus's pointer positions, in the former case the controlling apparatus has an effect on the quantum state of the system. Whereas the philosophical problems of understanding measurements in quantum mechanics are often discussed in the literature, the philosophical problems of the reverse process have hardly been discussed so far.

The symmetry between both problems can be described as follows: a measurement can be described as unitary operations on the apparatus conditioned on the system state, whereas unitary quantum state control can be described as unitary operation conditioned on the state (the 'adjustment') of the controller.

The only asymmetry is the fact that a measurement must *generate* entanglement for a generic system state whereas unitary control must *avoid* entanglement with the controller.

In our model the analogy as well as the difference of the two problems emerge rather clearly. We start by introducing a rather tight definition of QND-measurement ('quantum non-demolition measurement'). Recall that a QND-measurement is usually defined as a measurement procedure which does not change the eigenstates of the measured observable and only destroys superpositions of eigenstates corresponding to different eigenvalues. In other words, a QND-measurement for the observable A is a process mapping the state $|\phi\rangle\otimes|\psi\rangle$ to the state $\sum_j|\phi_j\rangle\otimes P_j|\psi\rangle$, where $|\phi\rangle$ is the initial state of the measurement apparatus and $|\phi_j\rangle$ are the final pointer states of the apparatus corresponding to the spectral projections P_j of A. They have to be orthogonal in order to enable a perfect distinction between the different measurement values. The notion of a QND-measurement does not include any assumption about the states of the total system *during* the time period $[0,T]$ between the beginning and the end of the procedure. In contrast, we define a CQND-measurement (*continuous quantum non-demolition measurement*) as a measuring procedure where the eigenstates of the measured observable are not changed at any moment *during* the measurement procedure. More precisely, we define:

Definition 2 An observable A is said to be **CQND-measurable** if the interface algebra $\mathcal{I}$ contains effective Hamiltonians $G_1,\ldots,G_l$ and 'times' $r_1,\ldots,r_l$ such that there exists an initial state $|\phi\rangle$ of the controller with the following properties:

- The concatenation of the corresponding dynamical evolutions $e^{iG_lr_l}\ldots e^{iG_1r_1}$ is a measurement in the sense explained in section 1
- The evolution does not disturb any eigenvector $|\psi\rangle$ of A during the measurement procedure, i.e, for every $j\leq l$ and $r\leq r_j$ we have

$$e^{iG_jr}e^{iG_{j-1}r_{j-1}}\ldots e^{iG_1r_1}(|\phi\rangle\otimes|\psi\rangle)=|\phi_j(r)\rangle\otimes|\psi\rangle \tag{3}$$

for some vector $|\phi_j(r)\rangle$.

The main purpose for introducing this definition is that it allows the use of powerful Lie-algebraic tools in the

sequel. We obtain necessary and sufficient conditions for the existence of CQND-procedures depending on the interface algebra. Obviously this implies necessary conditions for existence of QND-measurements in the usual sense.

We will show that the ability for performing CQND-measurements of the observable A is equivalent to the ability to implement the unitary one-parameter group $(e^{iAs})_{s\in\mathbb{R}}$. We define what we mean by the latter:

Definition 3 Let A be a self-adjoint operator on $\mathcal{H}_s$. The interaction Hamiltonian H is said to **enable the implementation of the group** $(e^{iAs})_{s\in\mathbb{R}}$ if the interface algebra $\mathcal{I}$ contains an element G and there is a state $|\phi\rangle$ of the controller such that

$$G(|\phi\rangle\otimes|\psi\rangle) = |\tilde{\phi}\rangle\otimes A|\psi\rangle \tag{4}$$

for an appropriate vector $|\tilde{\phi}\rangle\in\mathcal{H}_c$.

Our main theorem on the set of possible implementations and the set of possible CQND-measurements is the following:

Theorem 3 Let $H := \sum_j A_j\otimes B_j$ be the interaction between a quantum system and its controller with A_j and B_j as in Theorem 1. Assume $dim(\mathcal{H}_c)\geq dim(\mathcal{H}_s)$. Let $\mathcal{B}$ be the C^*-algebra generated by the operators B_j. If A is a self-adjoint operator acting on $\mathcal{H}_s$ the following statements are equivalent:

1. The interaction H enables a CQND-measurement of A
2. The interaction H enables the implementation of the one-parameter group $(e^{iAs})_{s\in\mathbb{R}}$
3. The operator A is an element of $\mathcal{B}$.

Proof: $3\Rightarrow 2$: Due to Theorem 1 the interface algebra $\mathcal{I}$ contains an operator of the form $E\otimes A$ with arbitrary self-adjoint $E\neq 0$. By initializing the controller to an eigenvector of E corresponding to an eigenvalue $\lambda\neq 0$ the evolution $e^{iE\otimes As}$ implements $e^{iA\lambda s}$.

$3\Rightarrow 1$: Let $(P_j)_{j\leq k}$ be the set of spectral projections of A. Since $\mathcal{B}$ is an algebra the operator $\sum_j jP_j$ is an element of $\mathcal{B}$. Due to Theorem 1 the Lie algebra $\mathcal{I}$ contains the operator $D\otimes\sum_j jP_j = \sum_j jD\otimes P_j$ with $D := diag(1,2,\ldots,n)-(n(n+1)/2)1$, where n is the dimension of $\mathcal{H}_c$. By initializing the controller in the state

$$|\phi\rangle := \frac{1}{\sqrt{n}}(1,1,\ldots,1)^T$$

the unitary operator

$$e^{i(\sum_j jD\otimes P_j)\pi/(2n)}$$

leads to mutually orthogonal final states of the controller for the states in the image of different spectral projections P_j: since $k\leq n$ (due to $dim(\mathcal{H}_c)\geq dim(\mathcal{H}_s)$) the vectors $e^{ijD\pi/(2n)}|\phi\rangle$ are mutually orthogonal for different $j=1,\ldots,k$.

$1\Rightarrow 3$: Choose operators G_j as in Definition 2. Define $|\phi_j(r)\rangle$ by the equation

$$e^{iG_jr}e^{iG_{j-1}r_{j-1}}\ldots e^{iG_1r_1}(|\phi\rangle\otimes|\psi\rangle) = |\phi_j(r)\rangle\otimes|\psi\rangle,$$

for $r\in[0,r_j]$ and every eigenstate $|\psi\rangle$ of A. Define Q_j to be the projection onto the span of the vectors $\{|\phi_j(r)\rangle\}_{r\in[0,r_j]}$. Due to Lemma 2 in [10] the operator

$$\tilde{G}_j := (Q_j\otimes 1)G_j(Q_j\otimes 1) - tr((Q_j\otimes 1)G_j(Q_j\otimes 1))1$$

is an element of $\mathcal{I}$. The operator $\tilde{G}_j$ does not change any eigenstate $|\psi\rangle$ of A. This can be seen as follows.

$$\tilde{G}_j|\phi_j(r)\rangle\otimes|\psi\rangle = \frac{d}{dr}|\phi_j(r)\rangle\otimes|\psi\rangle.$$

Since $\tilde{G}_j$ annihilates every vector in the complement of the span of $\{|\phi_j(r)\}$, every vector in the image of $\tilde{G}_j$ is a product vector with $|\psi\rangle$ as its second component.

Therefore $\tilde{G}_j$ can be written as a Hamiltonian on $\mathcal{H}_c$ which is conditioned on the eigenvalue of A, i.e., it has the form

$$\tilde{G}_j := \sum_k K_k^{(j)}\otimes P_k,$$

where $K_k^{(j)}$ are appropriate self-adjoint operators and P_k are the spectral projections of A. Since system states lying in the image of different projections P_k lead to orthogonal pointer states by assumption, for every pair k,m with $k\neq m$ there is a j such that $K_k^{(j)}\neq K_m^{(j)}(mod\mathbb{R}1)$. Hence there is a linear combination $\tilde{G}$ of the operators $\tilde{G}_j$ such that

$$\tilde{G} = M_k\otimes P_k$$

with $M_k\neq M_m$ $(mod\ \mathbb{R}1)$ for $k\neq m$. Using the main result of [10] we conclude that $\mathcal{I}$ contains every operators of the form $\sum_k kD\otimes P_k = \sum_k D\otimes kP_k$ with arbitrary traceless self-adjoint D. Due to Theorem 1 the operator $\sum_k kP_k$ is an element of $\mathcal{B}$. Since $\mathcal{B}$ is an algebra, A is an element of $\mathcal{B}$ too.

$2\Rightarrow 3$: Define P as the projection onto the span of $|\phi\rangle$ and $|\tilde{\phi}\rangle$ defined as in equation (4). Then $\tilde{G} := (P\otimes 1)G(P\otimes 1)$ is a nonzero operator of the form $D\otimes A$ an with appropriate operator D. Since $\tilde{G}$ is an element of the interface algebra, the operator A has to be an element of $\mathcal{B}$. □

The connection between measurements and implementations can be made even stronger in the special case that

A is an observable with equidistant eigenvalues. Consider the operator $G := D \otimes A$ where D is a operator with equidistant eigenvalues as well. Then the same effective Hamiltonian G allows the *implementation of* e^{iAs} by initializing the controller to an eigenstate of D with nonzero eigenvalue or *measurements of* A by taking the initial state $(1, \ldots, 1)^T$ for the controller.

V. THE ALGEBRAIC STRUCTURE OF THE SET OF MEASURABLE OBSERVABLES

The constructive part of the proof of Theorem 1 gives insights to a question which has been discussed since the early days of quantum mechanics, namely the algebraic structure of the set of observables. For non-commuting observables A and B the spectral projections of $A + B$ are not related to those of A and B in any obvious way. Hence there is a priori no obvious connection between measurement procedures for A and B and a procedure measuring $A + B$. Similarly, there is no obvious operational meaning for the *Jordan* product $(AB+BA)/2$ and the commutator $i[A, B]$. P. Jordan emphasized [14] that the equation

$$AB + BA = (A + B)^2 - A^2 - B^2$$

can reduce the operational meaning of the Jordan product to the operational meaning of the addition of observables since squaring of an observable can be interpreted as renaming its eigenvalues. He argued that the latter meaning is given by the fact that the expectation value of $A + B$ is the sum of the expectation values of A and B. Since we are interested only in measurements which project the state onto the eigenspaces, we are not content with this explanation. In contrast, we argue that the *algebraic* structure of the measurable observables is an implication of the *Lie algebraic* structure of $\mathcal{I}$. The latter structure has a direct operational meaning by its obvious connection to the sequences of local operations required for performing measurements.

In order to show, that measurements of algebraic expressions in A an B are related to measurements of A and B, we are not allowed to identify measurements of A and $f(A)$ for an arbitrary bijective function f on the spectrum of A. Note that we constructed measurements (in the proof of Theorem 3) of A by simulating effective Hamiltonians of the form $D \otimes f(A)$ where D and $f(A)$ is an observable with equidistant eigenvalues. In the following we assume a measurement procedure of any observable A to be implemented by an effective Hamiltonian of the form

$$E \otimes A,$$

with arbitrary self-adjoint E. This assumption can be interpreted as follows: If the system is in an eigenstate of A corresponding to the eigenvalue λ then the pointer of the measurement apparatus is moved due to the unitary evolution corresponding to the Hamiltonian λE, i.e., the pointer is moved in the same way having a velocity proportional to the eigenvalue λ. Note that (in the generic case) such an effective Hamiltonian will not lead to mutual orthogonal states of the controller for different eigenvalues λ. There is a pragmatic solution of this problem: if the dimension of $\mathcal{H}_c$ is large there are many examples of Hamiltonians E and initial states $|\phi\rangle$ such that the states $e^{i\lambda Es}|\phi\rangle$ are *almost* orthogonal for different eigenvalues λ. This can be illustrated by taking an infinite dimensional space $\mathcal{H}_c := L^2(\mathbb{R})$, i.e. the set of square integrable functions on the real line and think of E as an operator with continuous spectrum. Assume e.g.

$$E := i\frac{d}{dx}$$

Take small wave packages as initial states $|\phi\rangle$ of the controller. Then the dynamics

$$e^{i\frac{d}{dx}\otimes As}$$

shifts the wave package of the controller by the amount λs if the system's state is an eigenstate corresponding to the eigenvalues λ. If the width of the wave package is smaller than s times the minimal distance of the eigenvalues of A, the different wave packages are almost orthogonal.

Assume now we have operations on the controller simulating the effective Hamiltonians

$$H_A := E \otimes A \quad \text{and} \quad H_B := F \otimes B \tag{5}$$

with arbitrary $E, F \in \mathcal{W}$.

Without knowing A and B itself[5], their corresponding 'measurement Hamiltonians' H_A and H_B can be used for constructing measurements for self-adjoint algebraic expressions in A and B:

- **measuring (A+B)**

  Choose $\mathbb{R}$-linear maps $L_A, L_B : \mathcal{W} \to \mathcal{W}$ such that $L_A(E) = D$ and $L_B(F) = D$ with $D := diag(1, \ldots, n) - \sum_{j \le n} j$. Lemma 2 in [10] provides general rules for simulating $\tilde{H}_A := (L_A \otimes id)(H_A)$ and $\tilde{H}_B := (L_B \otimes id)(H_B)$ using the time evolutions due to H_A and H_B. By alternating the dynamical evolution due to the Hamiltonians $\tilde{H}_A$ and

[5]Note that the following schemes assume that the experimentalist knows E and F, but he needs not know the Hamiltonians H_A and H_B completely.

$\tilde{H}_B$ in small time steps we obtain an approximative simulation of the time evolution corresponding to $D \otimes (A + B)$. If the dimension n of the controller is large enough, there is a parameter $s \in \mathbb{R}^+$ such that the unitaries $e^{i\lambda_j Ds}$ lead to mutual orthogonal states for different eigenvalues λ_j of $A + B$ if the controller is initialized to the state $(1/\sqrt{n})(1, \ldots, 1)^T$.

- **measuring i[A,B]**

  Apply small steps of time evolutions generated by the effective Hamiltonians $D \otimes A$ and $D \otimes B$ followed by a simulation of the evolution due to $-D \otimes A$ and $-D \otimes A$ (note that the linear map $D \mapsto -D$ can be implemented by operations on the controller). This simulates an evolution corresponding to the effective Hamiltonian $D \otimes i[A, B]$ approximatively. As above, this can be used for measuring $i[A, B]$.

- **measuring (AB+BA)**

  Assume that E and F in eq. (5) do not commute (otherwise one can obtain a different effective Hamiltonian $\hat{H}_B$ by additional operations on the controller). The equation

  $$i[E \otimes A, F \otimes B] + i[E \otimes B, F \otimes A] = i[E, F] \otimes (AB + BA)$$

  provides a scheme for measuring $AB + BA$ as follows: simulate the Hamiltonians $\tilde{H}_B := E \otimes B$ and $\tilde{H}_A := F \otimes A$ by using operations on the controller which exchange E and F. Simulate $i[E, F] \otimes (AB + BA)$ by evolutions corresponding to $H_A, H_B, \tilde{H}_A, \tilde{H}_B$.

We do not expect, that the procedures presented above are the best ones for practical purposes. Our theory should rather illustrate that the algebraic structure of the set of observables can be understood by analyzing the continuous dynamics of the measurement procedure.

VI. CONCLUSIONS

We have shown that universal quantum state control does neither require the possibility of changing the Hamiltonian of the system to be controlled nor the ability to access its interaction Hamiltonian to the controller. We gave necessary and sufficient conditions on the interaction Hamiltonian to enable universal control by local quantum operations on the controller only. Although the 'true Hamiltonian' remains constant the experimentalists is able to simulate arbitrary Hamiltonian time evolutions of the controlled system. Hence our investigations should be considered as a model for 'designing' effective Hamiltonians although we can not change the true ones. We could show how measurement procedures for $A + B$, for $i[A, B]$ and $AB + BA$ are related to measurement procedures of A and B. This gives new insights to the algebraic structure of the set of quantum observables. We have shown that the *algebraic* structure can be derived from the *Lie* algebraic structure of the infinitesimal transformations on the joint system given by the measurement apparatus and the measured system.

Furthermore we could show in which way the quantum state of a spin chain can in principle be controlled by accessing only the first site. This should serve as a more general model for indirect quantum control.

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